stochastic processes and their

Self-intersection local time of an $\mathscr{S}'(\mathbb{R}^d)$-valued process involving motions of two types

Luis G. Gorostiza[a,*,1], Ekaterina Todorova[a,b,1]

[a] *Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados, A.P. 14-740, México 07000 D.F., Mexico*

[b] *Centro de Investigación en Matemáticas, A.C., A.P. 402, 36000 Guanajuato, Gto., Mexico*

Received 4 February 1998; accepted 18 December 1998

Abstract

We study existence and continuity of self-intersection local time (SILT) for a Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued process which arises as a high-density fluctuation limit of a particle system in $\mathbb{R}^d$, where the particle motion switches back and forth between symmetric stable processes of indices α_1 and α_2 at exponential time intervals. We prove that SILT exists if and only if $d < 2\min\{\alpha_1, \alpha_2\}$. This means that existence of SILT is determined by the "most mobile" of the two types, and we interpret this result in terms of the particle picture. In contrast with the single-type case, there are technical difficulties due to the lack of self-similarity of the particle paths. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Self-intersection local time; Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued process; Stable process

1. Introduction

Let $X = (X(t))_{t\in[0,1]}$ be a continuous centered Gaussian process with values in $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions on $\mathbb{R}^d$. An intuitive definition of the self-intersection local time (SILT) of X up to time $t \in [0,1]$ is given by the formal expression

$$\int_0^t \int_0^t \langle X(s) \otimes X(r), \delta(x-y)\varphi(x)\rangle \,\mathrm{d}s\,\mathrm{d}r, \tag{1.1}$$

where $\otimes$ denotes the tensor product in $\mathscr{S}'(\mathbb{R}^d)$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$ (the C^∞ rapidly decreasing functions), δ is the Dirac distribution, and $\langle\cdot,\cdot\rangle$ stands for duality. Since (1.1) does not make sense because $\delta(x-y)\varphi(x) \notin \mathscr{S}(\mathbb{R}^{2d})$, the first problem is how to give a rigorous meaning to it.

This problem was studied by Adler et al. (1991), and Adler and Rosen (1993) in the special case where X is the "α-stable density process" (with $\alpha = 2$, the Brownian case,

* Corresponding author.

E-mail address: gortega@servidor.unam.mx (L.G. Gorostiza)

[1] Research supported in part by CONACyT grant 211085-1317PE, Mexico.

0304-4149/99/$ – see front matter
PII: S0304-4149(98)00108-2

in Adler et al., 1991). This process is a high-density fluctuation limit of a Poisson system of independent particles undergoing symmetric α-stable motion. The particle system is called the "particle picture" for the process X. In Adler et al. (1991), and Adler and Rosen (1993) it is shown, among other things, that SILT exists for X if and only if $d < 2\alpha$, and for $\alpha = 2$ the SILT process, when it exists, has cadlag paths. Their methods rely partly on the particle picture, which is also used in Adler et al. (1991) to provide a justification for considering (1.1) as a SILT of X. This justification is based on the fact that the SILT of X is also obtained as a limit of intersection local times of all the pairs of particle paths, and that two independent symmetric α-stable paths in $\mathbb{R}^d$ intersect if and only if $d < 2\alpha$.

An existence and continuity criterion for SILT for a general class of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes was obtained by Bojdecki and Gorostiza (1995) (Theorem 2.2 below). This criterion is inspired in ideas of Adler et al. (1991), and Adler and Rosen (1993), but it does not rely on particle pictures (which in general do not exist). Since the rigorous definition of SILT (Definition 2.1 below) is rather abstract, it is not clear what properties of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes are characterized by SILT results. We recall that SILT for $\mathscr{S}'(\mathbb{R}^d)$-processes is not an extension of the concept in finite dimension (Bojdecki and Gorostiza, 1995). In the case of the α-stable density process, which has a simple associated particle picture, we have the interpretation given in Adler et al. (1995).

With a view toward increasing our understanding of the meaning of SILT for Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes which have associated particle pictures, in this paper we study existence and continuity of SILT for a process X which is also a high-density fluctuation limit of a particle system, but now the particle motion switches back and forth between two symmetric stable processes with different indices, α_1 and α_2, with exponential waiting times between changes. We show that SILT exists if and only if $d < 2\min\{\alpha_1, \alpha_2\}$ and SILT paths are continuous (Theorem 4.1). A particle picture interpretation of this result in the spirit of Adler et al. (1991) is of interest because for two different stable motions there are several possibilities of intersections depending on the dimension. The interpretation is that existence of SILT for X is determined by existence of intersections for pairs of paths of the "most mobile" type. This sheds some light into the meaning of SILT for Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes with associated particle pictures.

Other questions on the meaning of SILT are brought up in Bojdecki and Gorostiza (1997) regarding other types of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes, not necessarily having particle pictures.

The proof of existence and continuity of SILT will be done by means of the criterion of Bojdecki and Gorostiza (1995). However, the presence of two different stability indices causes technical difficulties which do not arise in the previous works on the subject. In Adler et al. (1995), Adler and Rosen (1993), and the various examples in Bojdecki and Gorostiza (1995) (see also Bojdecki and Gorostiza, 1996, 1997) the particle motion (a single symmetric α-stable process) is Markovian, and the proofs employ in a direct way the Chapman–Kolmogorov formula and the self-similarity of the motion. In our case we cannot follow the same line of proof because the particle motion is not Markovian and not self-similar. We proceed in two steps. First, we work under the assumption that the Markov chain of types (switching between α_1 and α_2)

is stationary, and then we treat the general case by means of a comparison with the stationary case through upper and lower bounds. In particular, the lack of self-similarity of the particle motion is dealt with by means of upper and lower bounds for the density. These calculations lead to show that the hypotheses of the criterion of Bojdecki and Gorostiza (1995) are satisfied.

In Section 2 we establish some notation and recall the SILT criterion from Bojdecki and Gorostiza (1995). In Section 3 we introduce the process X whose SILT we shall study by showing how it arises as a fluctuation limit. In Section 4 we state the theorem on SILT for the process X and we interpret the existence result in terms of the particle picture, and in Section 5 we prove the theorem.

2. Existence and continuity criterion for SILT of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes

We will condense the main result from Bojdecki and Gorostiza (1995). We refer the reader to that paper for additional information and details.

Let $\mathscr{F}$ denote the class of non-negative symmetric C^∞ functions f on $\mathbb{R}^d$ with bounded support, and such that $f(0)>0$ and $\int f(x)\,\mathrm{d}x=1$. For $f\in\mathscr{F}$, $\epsilon>0$ and $\varphi\in\mathscr{S}(\mathbb{R}^d)$ we define

$$f_\epsilon(x)=\epsilon^{-d}f\left(\frac{x}{\epsilon}\right),\quad x\in\mathbb{R}^d$$

and

$$\Phi^f_{\epsilon,\varphi}(x,y)=f_\epsilon(x-y)\varphi(x),\quad x,y\in\mathbb{R}^d.$$

Note that $\Phi^f_{\epsilon,\varphi}\in\mathscr{S}(\mathbb{R}^{2d})$ and $\Phi^f_{\epsilon,\varphi}$ approximates $\delta(x-y)\varphi(x)$ as $\epsilon\to 0$. In order to give a meaning to (1.1), the idea is to replace $\delta(x-y)\varphi(x)$ by $\Phi^f_{\epsilon,\varphi}$, so that it makes sense, and take the limit as $\epsilon\to 0$. But this is not enough; for existence of a limit it is also necessary to replace $X(s)\otimes X(r)$ by the Wick product $:X(s)\otimes X(r):$, which is an $\mathscr{S}'(\mathbb{R}^{2d})$-valued random field such that

$$\langle :X(s)\otimes X(r):,\varphi\otimes\psi\rangle=\langle X(s),\varphi\rangle\langle X(r),\psi\rangle-E(\langle X(s),\varphi\rangle\langle X(r),\psi\rangle).$$

This leads to defining an approximate SILT $L^f_\epsilon(t)$ by

$$\langle L^f_\epsilon(t),\varphi\rangle=\int_0^t\int_0^t\langle :X(s)\otimes X(r):,\Phi^f_{\epsilon,\varphi}\rangle\,\mathrm{d}s\,\mathrm{d}r,\quad t\in[0,1],\ \varphi\in\mathscr{S}(\mathbb{R}^d),$$

which is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process.

We can now give a precise meaning to (1.1).

Definition 2.1. For a given continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued process $X=(X(t))_{t\in[0,1]}$, if there exists an $\mathscr{S}'(\mathbb{R}^d)$-valued process $L=(L(t))_{t\in[0,1]}$ such that for any $t\in[0,1]$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$ and $f\in\mathscr{F}$,

$$\langle L^f_\epsilon(t),\varphi\rangle\to\langle L(t),\varphi\rangle$$

in L^2 as $\epsilon\to 0$, then L is called the *self-intersection local time* (SILT) of X.

Let K denote the covariance functional of X:

$$K(s,\varphi;t,\psi)=E(\langle X(s),\varphi\rangle\langle X(t),\psi\rangle),\quad \varphi,\psi\in\mathscr{S}(\mathbb{R}^d),\ s,t\in[0,1].$$

For test functions $\Phi^{(1)},\Phi^{(2)}\in\mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(\mathbb{R}^d)$ of the form

$$\Phi^{(1)}=\sum_{i=1}^{n}\varphi_i^{(1)}\otimes\psi_i^{(1)},\qquad \Phi^{(2)}=\sum_{j=1}^{m}\varphi_j^{(2)}\otimes\psi_j^{(2)},\varphi_i^{(1)},\psi_i^{(1)},\varphi_j^{(2)},\psi_j^{(2)}\in\mathscr{S}(\mathbb{R}^d) \tag{2.1}$$

and $s,r,u,v\in[0,1]$, we consider the functional

$$\begin{aligned}J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})=\sum_{i=1}^{n}\sum_{j=1}^{m}&(K(s,\varphi_i^{(1)};u,\varphi_j^{(2)})K(r,\psi_i^{(1)};v,\psi_j^{(2)})\\&+K(s,\varphi_i^{(1)};v,\psi_j^{(2)})K(r,\psi_i^{(1)};u,\varphi_j^{(2)})),\end{aligned} \tag{2.2}$$

which is the covariance functional of the random field $:X(s)\otimes X(r):$ for test functions of the given form.

We now state the existence and continuity criterion for SILT:

Theorem 2.2. (1) *Given a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X, assume that $J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$ has a well-defined extension on $\mathscr{S}(\mathbb{R}^{2d})\times\mathscr{S}(\mathbb{R}^{2d})$ such that*

(i) *The functional*

$$(\Phi^{(1)},\Phi^{(2)})\mapsto\int_{[0,t]^4}J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v$$

is continuous on $\mathscr{S}(\mathbb{R}^{2d})\times\mathscr{S}(\mathbb{R}^{2d})$ for each $t\in[0,1]$.

(ii) *$J_{s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})$ converges to a finite limit as $\epsilon,\delta\to 0$, for each $f,g\in\mathscr{F}$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$, $s,r,u,v\in[0,1]$, and this limit does not depend on f,g.*

(iii)

$$|J_{s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})|\leqslant G_\varphi(s,r,u,v)$$

for some measurable function G_φ on $[0,1]^4$ which depends on φ but is independent of ϵ,δ,f,g, and such that

$$\int_{[0,1]^4}G_\varphi(s,r,u,v)\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v<\infty$$

for each $\varphi\in\mathscr{S}(\mathbb{R}^d)$.

Then the SILT L of the process X exists.

Assume in addition that

(iv) *There exists a non-decreasing continuous function F on $[0,1]$ and a number $\gamma>0$ such that for all $t_1,t_2\in[0,1]$, $t_1<t_2$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$,*

$$\begin{aligned}&\int_{[0,1]^4}(1_{[0,t_2]^2}(s,r)-1_{[0,t_1]^2}(s,r))(1_{[0,t_2]^2}(u,v)-1_{[0,t_1]^2}(u,v))G_\varphi(s,r,u,v)\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\\&\quad\leqslant C(\varphi)(F(t_2)-F(t_1))^{1+\gamma},\end{aligned}$$

where $C(\varphi)$ is a positive constant depending only on φ.

Then the SILT L is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process, and moreover L_ϵ^f converges weakly to L in $C([0,1],\mathscr{S}'(\mathbb{R}^d))$ as $\epsilon \to 0$.

(2) *Suppose that $J_{s,r,u,v}$ satisfies condition* (i) *but*

$$\lim_{\epsilon\to 0}\int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v=\infty$$

for some $\varphi\in\mathscr{S}(\mathbb{R}^d)$ and $f\in\mathscr{F}$. Then X does not have SILT.

In applications, this criterion is employed to find the range of space dimensions d for which a given Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X has SILT, and to study if the SILT process is continuous. Note that particle pictures are not involved here.

3. The density process of a two-type particle system

Consider the following two-type particle system in $\mathbb{R}^d$. For each $i=1,2$, particles of type i move according to a symmetric α_i-stable process, and each particle (independently) switches back and forth between the two types with respective exponential waiting times with parameters V_i. At time 0 the particles of type i are distributed according to a Poisson random measure on $\mathbb{R}^d$ with intensity measure $n\gamma_i\lambda$, where $n>0$, $\gamma_i\geqslant 0$, and λ is the Lebesgue measure on $\mathbb{R}^d$. Thus we have a Poisson system of independent particles which evolve in $\mathbb{R}^d$ according to the "basic process", i.e. the process which switches stability indices at exponential time intervals.

Let

$$N_i^n(t)=\sum_j \delta_{\xi_j^i(t)},\quad i=1,2,$$

where $\{\xi_j^i(t)\}_j$ are the positions of the particles of type $i=1,2$, at time t, and consider the normalized fluctuation process $M^n=(M_1^n,M_2^n)$ defined by

$$M_i^n(t)=n^{-1/2}(N_i(t)-EN_i^n(t)),\quad t\in[0,1],\ i=1,2.$$

Note that M^n takes values in $(\mathscr{S}'(\mathbb{R}^d))^2\cong\mathscr{S}'(\mathbb{R}^{2d})$.

We will use the notation

$$\Phi\odot\Psi=(\varphi_1\psi_1,\varphi_2\psi_2)$$

if $\Phi=(\varphi_1,\varphi_2),\Psi=(\psi_1,\psi_2),\phi_i,\psi_i\in\mathscr{S}(\mathbb{R}^d)$, and

$$\langle\Gamma,\Phi\rangle=\sum_{i=1}^2\gamma_i\int\varphi_i(x)\,\mathrm{d}x,$$

where $\Gamma=(\gamma_1\lambda,\gamma_2\lambda)$.

The following result is a special case of Theorem 4.2 in López-Mimbela (1992).

Theorem 3.1. *M^n converges weakly to the process $M=(M_1,M_2)$ in $C([0,1],\mathscr{S}'(\mathbb{R}^{2d}))$ as $n\to\infty$, where M is a continuous centered Gaussian Markov process with covariance functional $K(s,\Phi;t,\Psi)=E(\langle M(s),\Phi\rangle\langle M(t),\Psi\rangle)$ given by*

$$K(s,\Phi;t,\Psi)=\langle\Gamma,U(s)(\Phi\odot U(t-s)\Psi)\rangle,\ s,t\in[0,1],\ s\leqslant t,\Phi,\Psi\in\mathscr{S}(\mathbb{R}^{2d})\quad(3.1)$$

and $(U(t))_{t\in[0,1]}$ is the semigroup of the basic (*position, type*) *process.*

The precise definition of $U(t)$ is given at the begining of Section 5.

Our objective is to study SILT for the following three Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes: M_1, M_2 and

$$X = M_1 + M_2. \tag{3.2}$$

Note that X is the density process of the Poisson system of basic processes. M_1 and M_2 are Markovian but X is not (the covariance of X does not satisfy $\mathrm{Cov}(\langle X_r, \varphi\rangle, \langle X_t, \psi\rangle) = \mathrm{Cov}(\langle X_r, \varphi\rangle, \langle X_{s,t}, \psi\rangle)$ for some $\mathscr{S}'(\mathbb{R}^d)$-random variable $X_{s,t}$, $r \leqslant s < t$, which is the condition that characterizes the Markov property for Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes: see Fernández (1991).

4. Self-intersection local time theorem and particle picture interpretation

Let M_1, M_2 and X be the continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes introduced in Section 3.

Theorem 4.1. (a) *If $d < 2\min\{\alpha_1, \alpha_2\}$, then the process X has SILT, and SILT is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process, and if $d \geqslant 2\min\{\alpha_1, \alpha_2\}$, then X does not have SILT.*

(b) *For each $i = 1, 2$, if $d < 2\alpha_i$, then the process M_i has SILT, and SILT is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process, and if $d \geqslant 2\alpha_i$, then M_i does not have SILT.*

Clearly, in the special case $\alpha_1 = \alpha_2 = \alpha$ the previous results reduce to the known one for the α-stable density process (Adler et al., 1991; Adler and Rosen, 1993; Bojdecki and Gorostiza, 1995).

Note that the constants $\gamma_1, \gamma_2, V_1, V_2$ do not appear in the SILT results, as it should be expected. Nevertheless, they are relevant for the generality of the particle model, and they play a role in the proofs.

The processes M_1 and M_2 are more complicated than the α-stable density process because they are fluctuation limits of particle systems where the particles appear and disappear at random times. Indeed, their covariances are not so simple as the one for the α-stable density process and therefore the proofs require more work. (The covariance of the α-stable density process is $\int \varphi(x) T_{t-s}\psi(x)\,\mathrm{d}x$, where T_t is the α-stable semigroup, and the covariances of M_1 and M_2 are given by (5.32) and (5.33) below.) Nevertheless, these two processes are not too different from the α-stable process because each one of them involves only particles of a single type. Hence, it is not surprising that the SILT results for these processes coincide with those of the α-stable density process. Moreover, existence of SILT for M_1 and M_2 can be interpreted in terms of the particle picture similarly as in Adler et al. (1991): SILT for M_i exists if and only if two independent α_i-stable processes intersect, i.e., if and only if $d < 2\alpha_i$.

For the process X the particle picture interpretation of existence is that SILT exists if and only if independent α_i-stable and α_j-stable processes intersect for all $i, j = 1, 2$. Indeed, such intersections occur if and only if $d < \alpha_i + \alpha_j$ (see Taylor, 1966, Lemma 16) for all $i, j = 1, 2$, i.e., if and only if $d < 2\min\{\alpha_1, \alpha_2\}$. Intuitively, one could think that SILT exists if $d < 2\max\{\alpha_1, \alpha_2\}$ because at any time there is a large density of $\max\{\alpha_1,\alpha_2\}$-particles, so that the set of their intersections, which exist for these dimensions, should be noticeable. The reason that this is not so is that there is also a large density of $\min\{\alpha_1,\alpha_2\}$-particles, and since these do not intersect for $d\geqslant 2\min\{\alpha_1,\alpha_2\}$, the L^2-limit which defines SILT (following the argument of Adler et al., 1991) does not exist.

The fact that the most mobile type has a determining property occurs also in another situation: persistence/extinction for a class of two-type branching particle systems (Gorostiza et al., 1992; López-Mimbela and Wakolbinger, 1996). However, this is a quite different phenomenon: persistence/extinction concerns long time behavior, whereas SILT is local in time.

5. Proof of the theorem

The proof of part (a) contains most of the technical work. Some of it will also be useful for part (b), which will only be outlined.

We begin with some notation. We will omit writing $\mathbb{R}^d$ in the integrals on $\mathbb{R}^d$.

We denote by $\xi(t)$ and $\tau(t)$ the position and the type of the basic process at time t, respectively. The process $(\xi,\tau)=(\xi(t),\ \tau(t))_{t\in[0,1]}$ is Markovian, and we designate its semigroup by $(U(t))_{t\in[0,1]}$ (with domain $\mathscr{S}(M^d)\times\mathscr{S}(M^d)$). Its infinitesimal generator is given by

$$\begin{pmatrix}\Delta_{\alpha_1} & 0\\ 0 & \Delta_{\alpha_2}\end{pmatrix}+\begin{pmatrix}-V_1 & V_1\\ V_2 & -V_2\end{pmatrix}, \tag{5.1}$$

where $\Delta_\alpha\equiv-(-\Delta)^{\alpha/2}$. For $U(t)$ acting on functions of the form $F(x,i)=\varphi(x)1_{\{j\}}(i)$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$, $x\in\mathbb{R}^d$, $i,j\in\{1,2\}$, we write

$$U_{ij}(t)\varphi(x)=U(t)F(x,i). \tag{5.2}$$

Thus,

$$U_{ij}(t)\varphi(x)=\int\varphi(y)h_t^{(ij)}(x,y)\,\mathrm{d}y, \tag{5.3}$$

where

$$\begin{aligned}h_t^{(ij)}(x,y)\,\mathrm{d}y=&P[\xi(t)\in\ \mathrm{d}y\mid\xi(0)=x,\ \tau(0)=i,\ \tau(t)=j]\\ &P[\tau(t)=j\mid\tau(0)=i].\end{aligned} \tag{5.4}$$

For $t>0$, the functions $h_t^{(ij)}(x,\cdot)$ are continuous and integrable, and they satisfy

$$h_t^{(ij)}(x,y)=h_t^{(ij)}(x-y)\quad\text{and}\quad\int h_t^{(ij)}(x)\,\mathrm{d}x\leqslant 1. \tag{5.5}$$

We define the operators

$$U_i(t)=\gamma_1U_{1i}(t)+\gamma_2U_{2i}(t),\quad i=1,2. \tag{5.6}$$

Hence,

$$U_i(t)\varphi(x)=\int\varphi(y)h_t^{(i)}(x,y)\,\mathrm{d}y,\quad i=1,2, \tag{5.7}$$

where

$$h_t^{(i)}(x,y) = \gamma_1 h_t^{(1i)}(x,y) + \gamma_2 h_t^{(2i)}(x,y), \quad i = 1,2. \tag{5.8}$$

For $t > 0$, the functions $h_t^{(i)}(x,\cdot)$ are continuous and integrable.

We also define $p_t^{(i)}(x,y)$, $i = 1,2$, by

$$p_t^{(i)}(x,y) = h_t^{(i1)}(x,y) + h_t^{(i2)}(x,y), \tag{5.9}$$

hence

$$(U_{i1}(t) + U_{i2}(t))\varphi(x) = \int \varphi(y) p_t^{(i)}(x,y)\,\mathrm{d}y. \tag{5.10}$$

For $t > 0$, the functions $p_t^{(i)}(x,\cdot)$ are continuous and integrable.

From (5.4) we have

$$\int \varphi(y) p_t^{(i)}(x,y)\,\mathrm{d}y = E[\varphi(\xi(t)) \mid \xi(0) = x, \tau(0) = i],$$

therefore, by a slight abuse of language we refer to $p_t^{(i)}(x,y)$ as the transition densities of the position component of the basic process.

Lemma 5.1. *The covariance functional of the process X defined by* (3.2) *is given by*

$$\begin{aligned} K_X(s,\varphi;t,\psi) &= E(\langle X(s),\varphi\rangle\langle X(t),\psi\rangle) \\ &= \sum_{i=1}^{2} \int U_i(s)\left[\varphi(\cdot)\int p_{t-s}^{(i)}(\cdot,y)\psi(y)\,\mathrm{d}y\right](x)\,\mathrm{d}x, \\ &\quad 0 \leqslant s \leqslant t, \ \varphi,\psi \in \mathscr{S}(\mathbb{R}^d), \end{aligned} \tag{5.11}$$

where U_i and $p^{(i)}$ are defined in (5.6) *and* (5.9).

In (5.11), U_i and $p^{(i)}$ can be written in terms of the $h^{(ij)}$, but the hybrid notation will be useful for calculations.

Proof. From (3.1) with $\Phi = (\varphi,\varphi)$, $\Psi = (\psi,\psi)$, and (5.2), (5.3), (5.10) we obtain

$$\begin{aligned} &K_X(s,\varphi;t,\psi) \\ &= \left\langle \begin{pmatrix} \gamma_1 \Lambda \\ \gamma_2 \Lambda \end{pmatrix}, \begin{pmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{pmatrix} \begin{pmatrix} \varphi(U_{11}(t-s)\psi + U_{12}(t-s)\psi) \\ \varphi(U_{21}(t-s)\psi + U_{22}(t-s)\psi) \end{pmatrix} \right\rangle \\ &= \gamma_1 \int (U_{11}(s)[\varphi(U_{11}(t-s)\psi + U_{12}(t-s)\psi)](x) \\ &\quad + U_{12}(s)[\varphi(U_{21}(t-s)\psi + U_{22}(t-s)\psi)](x))\,\mathrm{d}x \\ &\quad + \gamma_2 \int (U_{21}(s)[\varphi(U_{11}(t-s)\psi + U_{12}(t-s)\psi)](x) \\ &\quad + U_{22}(s)[\varphi(U_{21}(t-s)\psi + U_{22}(t-s)\psi)](x))\,\mathrm{d}x \end{aligned}$$

$$= \gamma_1 \int U_{11}(s) \left(\varphi(x) \int p_{t-s}^{(1)}(x, y) \psi(y) \, \mathrm{d}y \right) \mathrm{d}x$$

$$+ \gamma_1 \int U_{12}(s) \left(\varphi(x) \int p_{t-s}^{(2)}(x, y) \psi(y) \, \mathrm{d}y \right) \mathrm{d}x$$

$$+ \gamma_2 \int U_{21}(s) \left(\varphi(x) \int p_{t-s}^{(1)}(x, y) \psi(y) \, \mathrm{d}y \right) \mathrm{d}x$$

$$+ \gamma_2 \int U_{22}(s) \left(\varphi(x) \int p_{t-s}^{(2)}(x, y) \psi(y) \, \mathrm{d}y \right) \mathrm{d}x,$$

where for simplicity we have written

$$U_{ij}(s) \left(\varphi(\cdot) \int p_{t-s}^{(j)}(\cdot, y) \varphi(y) \, \mathrm{d}y \right)(x) = U_{ij}(s) \left(\varphi(x) \int p_{t-s}^{(j)}(x, y) \varphi(y) \, \mathrm{d}y \right).$$

An additional step using (5.6) yields (5.11). □

The invariant measure

$$\overline{\Gamma} = (\overline{\gamma}_1 \lambda, \overline{\gamma}_1 \lambda) \tag{5.12}$$

for the Markov chain of types τ, which makes it stationary, is given by the constants

$$\overline{\gamma}_1 = \frac{V_2}{V_1 + V_2}, \qquad \overline{\gamma}_2 = \frac{V_1}{V_1 + V_2} \tag{5.13}$$

and $\overline{\Gamma}$ is invariant for the semigroup $U(t)$ (note that λ is invariant for the stable transition densities).

In the stationary case we have:

Corollary 5.2. *For $\Gamma = \overline{\Gamma}$, the covariance $\overline{K} \equiv K_X$ is given by*

$$\overline{K}(s, \varphi; t, \psi) = \langle \overline{\Gamma}, \Phi \odot U(t-s)\Psi \rangle = \sum_{i=1}^{2} \overline{\gamma}_i \int \varphi(x) \left(\int p_{t-s}^{(i)}(x, y) \psi(y) \, \mathrm{d}y \right) \mathrm{d}x,$$

$$0 \leqslant s \leqslant t, \Phi = (\varphi, \varphi), \Psi = (\psi, \psi), \varphi, \psi \in \mathscr{S}(\mathbb{R}^d).$$

The functional J in (2.2) takes the form in the next lemma. We omit the proof because it involves long but straightforward calculations.

Lemma 5.3. *For functions of the form* (2.1) *and $s \leqslant r \leqslant u \leqslant v$, the functional $J_{s,r,u,v}$ is given by*

$$J_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) = J_{s,r,u,v}^{(1)}(\Phi^{(1)}, \Phi^{(2)}) + J_{s,r,u,v}^{(2)}(\Phi^{(1)}, \Phi^{(2)}) \tag{5.14}$$

with

$$J_{s,r,u,v}^{(1)}(\Phi^{(1)}, \Phi^{(2)})$$

$$= \sum_{i,j=1}^{2} \int U_i(s) U_j(r) \left(\int \Phi^{(1)}(x, z) \Phi^{(2)}(y, w) p_{u-s}^{(i)}(x, y) p_{v-r}^{(j)}(z, w) \, \mathrm{d}y \, \mathrm{d}w \right) \mathrm{d}x \, \mathrm{d}z \tag{5.15}$$

and

$$J^{(2)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$$
$$=\sum_{i,j=1}^{2}\int U_i(s)U_j(r)\left(\int \Phi^{(1)}(x,z)\Phi^{(2)}(w,y)p^{(i)}_{v-s}(x,y)p^{(j)}_{u-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}x\,\mathrm{d}z, \tag{5.16}$$

where the operators $U_i(s)$ act on x and the operators $U_j(r)$ act on z.

In the stationary case we have:

Corollary 5.4. *For $\Gamma=\overline{\Gamma}$, $\overline{J}\equiv J$ is given by*

$$\overline{J}=\overline{J}^{(1)}+\overline{J}^{(2)}, \tag{5.17}$$

where

$$\overline{J}^{(1)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})=\sum_{i,j=1}^{2}\overline{\gamma}_i\overline{\gamma}_j\int \Phi^{(1)}(x,z)\Phi^{(2)}(y,w)p^{(i)}_{u-s}(x,y)p^{(j)}_{v-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}z \tag{5.18}$$

and

$$\overline{J}^{(2)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})=\sum_{i,j=1}^{2}\overline{\gamma}_i\overline{\gamma}_j\int \Phi^{(1)}(x,z)\Phi^{(2)}(w,y)p^{(i)}_{v-s}(x,y)p^{(j)}_{u-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}z. \tag{5.19}$$

Definition 5.5. We define the following bilinear functionals on $\mathscr{S}(\mathbb{R}^{2d})\times\mathscr{S}(\mathbb{R}^{2d})$:

$$H^{(1,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$$
$$=\int U_i(u)U_j(v)\left(\int \Phi^{(1)}(x,z)\Phi^{(2)}(y,w)p^{(i)}_{s}(x,y)p^{(j)}_{r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}x\,\mathrm{d}z, \tag{5.20}$$

$$H^{(2,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$$
$$=\int U_i(u)U_j(v)\left(\int \Phi^{(1)}(x,z)\Phi^{(2)}(w,y)p^{(i)}_{s}(x,y)p^{(j)}_{r}(z,w)\,\mathrm{d}y\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}w \tag{5.21}$$

for $s,r,u,v\in[0,1], i,j=1,2$, where $U_i(u)$ acts on x and $U_j(v)$ acts on z.

Lemma 5.6. *The functionals $H^{(1,i,j)}_{s,r,u,v}$ and $H^{(2,i,j)}_{s,r,u,v}$ of Definition 5.5 are continuous.*

Proof. Assume $s,r>0$ (the proof is similar for $s=0$ or $r=0$).

The operators $U_i(u)$ are bounded on $L^1(\mathbb{R}^d)$ and positive. Hence, from (5.20) we have

$$|H^{(1,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})|$$
$$\leqslant\int U_i(u)U_j(v)\left(\int |\Phi^{(1)}(x,z)||\Phi^{(2)}(y,w)|\,p^{(i)}_{s}(x,y)p^{(j)}_{r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}x\,\mathrm{d}z$$

$$\leqslant \sup_{y,w\in R^d} |\Phi^{(2)}(y,w)| \int U_i(u)U_j(v) \left[|\Phi^{(1)}(x,z)| \left(\int p_s^{(i)}(x,y)\,\mathrm{d}y \right) \right.$$

$$\left. \times \left(\int p_r^{(j)}(z,w)\,\mathrm{d}w \right) \right] \mathrm{d}x\,\mathrm{d}z$$

$$= c_1 \sup\nolimits_{y,w\in R^d} |\Phi^{(2)}(y,w)| \int U_i(u)U_j(v) \left(|\Phi^{(1)}(x,z)| \right) \mathrm{d}x\,\mathrm{d}z$$

$$\leqslant c_2 \sup_{y,w\in R^d} |\Phi^{(2)}(y,w)| \int |\Phi^{(1)}(x,z)|\,\mathrm{d}x\,\mathrm{d}z,$$

where c_1 and c_2 are positive constants. Hence, $H^{(1,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$ is separately continuous in $\Phi^{(1)}$ and $\Phi^{(2)}$. But a separately continuous bilinear functional on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ is (jointly) continuous (see Treves, 1967; Corollary to Theorem 34.1).

The same proof holds for $H^{(2,i,j)}_{s,r,u,v}$. □

The following corollary shows that condition (i) of Theorem 2.2 is satisfied.

Corollary 5.7. *The functional $J_{s,r,u,v}$ given by* (5.14)–(5.16) *has a well-defined extension on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ and*

$$(\Phi^{(1)},\Phi^{(2)}) \mapsto \int_{[0,t]^4} J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v$$

is continuous on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ for each $t \in [0,1]$.

Proof. From the proof of Lemma 5.6 we have, for a positive integer k,

$$|H^{(1,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})| \leqslant c_1 \sup_{y,w\in R^d} |\Phi^{(2)}(y,w)| \sup_{x,z\in R^d} (1+|x|^2)^k (1+|z|^2)^k |\Phi^{(1)}(x,z)| K,$$

where

$$K = \sup_{u,v\in[0,1]} \int U_i(u)U_j(v)(1+|x|^2)^{-k}(1+|z|^2)^{-k}\,\mathrm{d}x\,\mathrm{d}z.$$

Taking k large enough so that $(1+|x|^2)^{-k} \in L^1(\mathbb{R}^d)$, and using the fact that $t \mapsto U_i(t)(1+|x|^2)^{-k}$ is a continuous curve in $L^1(\mathbb{R}^d)$ (also for U_j) (Pazy, 1983, Corollary 2.3), we have that $K < \infty$. The same argument in the proof of Lemma 5.6 shows that

$$(\Phi^{(1)},\Phi^{(2)}) \mapsto \int_{[0,t]^4} |H^{(1,i,j)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})|\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v$$

is continuous.

Similarly for $H^{(2,i,j)}_{s,r,u,v}$, and the result follows from (5.14)–(5.16). □

The next two lemmas contain results on the transition densities of the α_i-stable processes and the basic process.

We designate by $q_t^{(i)}(x,y)$ the transition density of the symmetric α_i-stable process, $i = 1,2$.

Lemma 5.8. *The transition densities $p_t^{(i)}(x,y)$ of the position component of the basic process satisfy the system of integral equations*

$$p_t^{(1)}(x,y) = \mathrm{e}^{-V_1 t} q_t^{(1)}(x,y) + V_1 \int_0^t \mathrm{e}^{-V_1 s} \int q_s^{(1)}(x,z) p_{t-s}^{(2)}(z,y)\,\mathrm{d}z\,\mathrm{d}s,$$

$$p_t^{(2)}(x,y) = \mathrm{e}^{-V_2 t} q_t^{(2)}(x,y) + V_2 \int_0^t \mathrm{e}^{-V_2 s} \int q_s^{(2)}(x,z) p_{t-s}^{(1)}(z,y)\,\mathrm{d}z\,\mathrm{d}s. \tag{5.22}$$

Proof. This is done by a usual renewal argument (see e.g. Athreya and Ney, 1972). □

We denote by $\theta^{(i)}(t)$ the total time in the interval $[0,t]$ during which the basic process moves according to α_i starting with type i, on the event that there is a change of type in $(0,t]$.

Lemma 5.9. (a) *The following bounds hold for the densities $p_t^{(i)}(x,y)$, $i = 1,2$, $t \in (0,1]$:*

$$p_t^{(i)}(x,y) \geqslant a q_t^{(i)}(x,y), \tag{5.23}$$

$$p_t^{(i)}(x,y) \leqslant b t^{-d/\min\{\alpha_1,\alpha_2\}} \tag{5.24}$$

for some positive constants a and b.

(b) $p_t^{(i)}(x,y) = p_t^{(i)}(x-y)$, $i = 1,2$.

(c) *For $i \neq j$,*

$$p_t^{(i)}(x,y) \leqslant q_t^{(i)}(x,y) + \int_0^t \int q_s^{(i)}(x,z) q_{t-s}^{(j)}(z,y)\,\mathrm{d}z\,\mu_t^{(i)}(\mathrm{d}s), \quad t \in [0,1],$$

where

$$\mu_t^{(i)}(\mathrm{d}s) = P(\theta_t^{(i)} \in \mathrm{d}s).$$

(d) *There is a positive constant $C = C(d,\alpha_1,\alpha_2)$ such that*

$$\sup_{x,y \in \mathbb{R}^d} \int q_r^{(1)}(x,z) q_s^{(2)}(z,y)\,\mathrm{d}z \leqslant C(r+s)^{-d/\min\{\alpha_1,\alpha_2\}}$$

for all $s,r \in (0,1]$.

Proof. Assume $\alpha_1 \leqslant \alpha_2$.

(a) Eq. (5.23) follows immediately from (5.22). We will prove (5.24). Consider $p_t^{(1)}$. Since the convolution of densities is commutative, we may suppose that on the event that there is a change of type in $(0,t]$, all the α_1-motions take place first, and then all the α_2-motions. Hence we can rewrite (5.22) in the form

$$\begin{aligned} p_t^{(1)}(x,y) &= \mathrm{e}^{-V_1 t} q_t^{(1)}(x,y) + \int_0^t \int q_s^{(1)}(x,z) q_{t-s}^{(2)}(z,y)\,\mathrm{d}z\,P(\theta_t^{(1)} \in \mathrm{d}s) \\ &= \mathrm{e}^{-V_1 t} q_t^{(1)}(x,y) + I_1 + I_2, \end{aligned} \tag{5.25}$$

where

$$I_1 = \int_0^{t/2} \int q_s^{(1)}(x,z) q_{t-s}^{(2)}(z,y)\, dz\, P(\theta_t^{(1)} \in ds)$$

and

$$I_2 = \int_{t/2}^{t} \int q_s^{(1)}(x,z) q_{t-s}^{(2)}(z,y)\, dz\, P(\theta_t^1 \in ds).$$

By the self-similarity of the α_i-stable process

$$q_t^{(i)}(x,y) \leqslant c_1 t^{-d/\alpha_i}, \quad x,y \in R^d, \ i=1,2.$$

Hence,

$$I_1 \leqslant \int_0^{t/2} \int q_s^{(1)}(x,z) c_1 (t-s)^{-d/\alpha_2}\, dz\, P(\theta_t^{(1)} \in ds)$$

$$\leqslant c_1 \int_0^{t/2} \int q_s^{(1)}(x,z)(t/2)^{-d/\alpha_1}\, dz\, P(\theta_t^{(1)} \in ds)$$

(since $t-s \geqslant t/2$ and $\alpha_1 \leqslant \alpha_2$)

$$\leqslant c_2 t^{-d/\alpha_1},$$

where c_1 and c_2 are some positive constants.

Similarly,

$$I_2 \leqslant \int_{t/2}^{t} \int c_1 s^{-d/\alpha_1} q_{t-s}^{(2)}(z,y)\, dz\, P(\theta_t^{(1)} \in ds) \leqslant c_2 t^{-d/\alpha_1}.$$

The result is proved for $p_t^{(1)}$, and the same proof holds for $p_t^{(2)}$.

(b) Using the fact that $q_t^{(i)}(x,y)$ is a function of $x-y$ we have, from (5.25),

$$\begin{aligned} p_t^{(1)}(x,y) &= e^{-V_1 t} q_t^{(1)}(x-y) + \int_0^t \int q_s^{(1)}(x,z) q_{t-s}^{(2)}(z,y)\, dz\, P(\theta_t^{(1)} \in ds) \\ &= e^{-V_1 t} q_t^{(1)}(x-y) + \int_0^t \int q_s^{(1)}(x-z) q_{t-s}^{(2)}(z-y)\, dz\, P(\theta_t^{(1)} \in ds) \\ &= e^{-V_1 t} q_t^{(1)}(x-y) + \int_0^t \int q_s^{(1)}(x-y-w) q_{t-s}^{(2)}(w)\, dw\, P(\theta_t^{(1)} \in ds), \end{aligned}$$

which is a function of $x-y$. Similarly for $p_t^{(2)}$.

(c) This is an obvious consequence of (5.25) for $p_t^{(1)}$. Similarly for $p_t^{(2)}$.

(d) The function

$$y \mapsto \int_{R^d} q_r^{(1)}(x,z) q_s^{(2)}(z,y)\, dz$$

is unimodal and has a maximum at $y=x$ (see e.g. Lukacs, 1970, p. 98). Hence, by Plancherel's theorem,

$$\begin{aligned} \int q_r^{(1)}(x,z) q_s^{(2)}(z,y)\, dz &\leqslant \int q_r^{(1)}(x,z) q_s^{(2)}(z,x)\, dz \\ &= \int q_r^{(1)}(z) q_s^{(2)}(z)\, dz \end{aligned}$$

$$= \int \mathrm{e}^{-|z|^{\alpha_1} r} \mathrm{e}^{-|z|^{\alpha_2} s} \, \mathrm{d}z$$

$$= \int_{|z|>1} \mathrm{e}^{-|z|^{\alpha_1} r} \mathrm{e}^{-|z|^{\alpha_2} s} \, \mathrm{d}z + \int_{|z|\leqslant 1} \mathrm{e}^{-|z|^{\alpha_1} r} \mathrm{e}^{-|z|^{\alpha_2} s} \, \mathrm{d}z$$

$$= I_1 + I_2.$$

For I_1, $|z|^{\alpha_1} \leqslant |z|^{\alpha_2}$, and making the change of variable $z = (r+s)^{-1/\alpha_1} w$, we have

$$I_1 \leqslant \int \mathrm{e}^{-|z|^{\alpha_1}(r+s)} \, \mathrm{d}z = C_1 (r+s)^{-d/\alpha_1},$$

where $C_1 = \int \mathrm{e}^{-|w|^{\alpha_1}} \, \mathrm{d}w$.

For I_2, $|z|^{\alpha_2} \leqslant |z|^{\alpha_1}$, and since $t^{-d/\alpha_2} \leqslant C(d, \alpha_1, \alpha_2) t^{-d/\alpha_1}$ for $t \leqslant 2$, making the change of variable $z = (r+s)^{-1/\alpha_2} w$ we have

$$I_2 \leqslant \int \mathrm{e}^{-|z|^{\alpha_2}(r+s)} \, \mathrm{d}z = C_2 (r+s)^{-d/\alpha_2} \leqslant C_3 (r+s)^{-d/\alpha_1},$$

where $C_2 = \int \mathrm{e}^{-|w|^{\alpha_2}} \, \mathrm{d}w$ and $C_3 = C_2 2^{d(1/\alpha_1 - 1/\alpha_2)}$.

Substituting above, we obtain the result. □

Remark. Inequality (5.24) was proved at an early stage when we thought it would be needed. It turned out that a less precise result is enough. However, we include this inequality because it has an independent interest regarding the distribution of the basic process.

We will now show that condition (ii) of Theorem 2.2 holds.

Proposition 5.10. *For* $J^{(i)}_{s,r,u,v}$, *given by* (5.15) *and* (5.16), *the limits*

$$\lim_{\epsilon,\delta\to 0} J^{(i)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi}), \quad i = 1,2$$

exist for all $f, g \in \mathscr{F}, \varphi \in \mathscr{S}(\mathbb{R}^d), s, r, u, v \in [0,1]$, *and they are independent of* f *and* g.

In the stationary case (5.18) *and* (5.19) *the limits are given by*

$$\lim_{\epsilon,\delta\to 0} \overline{J}^{(1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi}) = \sum_{i,j=1}^{2} \int \varphi(x)\varphi(y) \overline{\gamma}_i \overline{\gamma}_j p^{(i)}_{u-s}(x,y) p^{(j)}_{v-r}(x,y) \, \mathrm{d}x \, \mathrm{d}y \quad (5.26)$$

and

$$\lim_{\epsilon,\delta\to 0} \overline{J}^{(2)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi}) = \sum_{i,j=1}^{2} \int \varphi(x)\varphi(y) \overline{\gamma}_i \overline{\gamma}_j p^{(i)}_{v-s}(x,y) p^{(j)}_{u-r}(x,y) \, \mathrm{d}x \, \mathrm{d}y. \quad (5.27)$$

Proof. We may assume that s, r, u, v are all different and $s < u, r < v$. The proof is analogous for the remaining cases.

Assume first $\Gamma = \overline{\Gamma}$. From (5.18),

$$\overline{J}^{(1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi}) = \sum_{i,j=1}^{2} \int \varphi(x) f_\epsilon(x-z)\varphi(y) g_\delta(y-w)\overline{\gamma}_i\overline{\gamma}_j p^{(i)}_{u-s}(x,y) p^{(j)}_{v-r}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w. \tag{5.28}$$

Let us consider the $i=j=1$ term. Since $f_\epsilon(x-z)g_\delta(y-w)$ is an approximation of the Dirac delta in $\mathbb{R}^{2d}$,

$$\lim_{\epsilon,\delta\to 0} \int f_\epsilon(x-z) g_\delta(y-w) p^{(1)}_{v-r}(z,w)\,\mathrm{d}z\,\mathrm{d}w = p^{(1)}_{v-r}(x,y).$$

Hence, by the Lebesgue's dominated convergence theorem,

$$\lim_{\epsilon,\delta\to 0} \overline{\gamma}_1^2 \int \varphi(x)\varphi(y) p^{(1)}_{u-s}(x,y)\left[\int f_\epsilon(x-z) g_\delta(y-w) p^{(1)}_{v-r}(z,w)\,\mathrm{d}z\,\mathrm{d}w\right]\mathrm{d}x\,\mathrm{d}y$$
$$= \overline{\gamma}_1^2 \int \varphi(x)\varphi(y) p^{(1)}_{u-s}(x,y) p^{(1)}_{v-r}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

We will justify the use of Lebesgue's theorem. Let

$$F_{\epsilon,\delta}(x,y) = \left|\int \varphi(x) f_\epsilon(x-z)\varphi(y) g_\delta(y-w) p^{(1)}_{u-s}(x,y) p^{(1)}_{v-r}(z,w)\,\mathrm{d}z\,\mathrm{d}w\right|.$$

By (5.24) $\sup_x p^{(1)}_{u-s}(x,y) \leqslant c$ and $\sup_y p^{(1)}_{v-r}(x,y) \leqslant c$ for some positive constant c; then

$$F_{\epsilon,\delta}(x,y) \leqslant c_1|\varphi(x)||\varphi(y)| \int f_\epsilon(x-z)\,\mathrm{d}z \int g_\delta(y-w)\,\mathrm{d}w$$
$$= c_1|\varphi(x)||\varphi(y)|$$

for some constant c_1. Hence $F_{\epsilon,\delta}$ is dominated by an integrable function.

Proceeding similarly with the other terms we obtain (5.26), and analogously (5.27).

Consider now an arbitrary intensity Γ, and denote by $J^{(1,1)}_{s,r,u,v}$ the $i=j=1$ term of $J^{(1)}_{s,r,u,v}$ in (5.15):

$$J^{(1,1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi}) = \int U_1(s)U_1(r)\left(\int \varphi(x) f_\epsilon(x-z)\varphi(y) g_\delta(y-w) \times p^{(1)}_{u-s}(x,y) p^{(1)}_{v-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}x\,\mathrm{d}z.$$

We have

$$\lim_{\epsilon,\delta\to 0} J^{(1,1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi})$$
$$= \int \lim\nolimits_{\epsilon,\delta\to 0} U_1(s)U_1(r)\left(\int \varphi(x) f_\epsilon(x-z)\varphi(y) g_\delta(y-w) \times p^{(1)}_{u-s}(x,y) p^{(1)}_{v-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}x\,\mathrm{d}z$$

$$= \int \lim_{\epsilon,\delta\to 0} U_1(s)U_1(r)\varphi(x)f_\epsilon(x-z)\int \varphi(y)g_\delta(y-w)$$
$$\times\, p^{(1)}_{u-s}(x,y)p^{(1)}_{v-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}z, \tag{5.29}$$

where Lebesgue's theorem was used for the first equality.

The justification for Lebesgue's theorem now is harder. With the notation of Definition 5.5 we have

$$H^{(1,1,1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})$$
$$=\int U_1(u)U_1(v)\varphi(x)f_\epsilon(x-z)\int \varphi(y)g_\delta(y-w)p^{(1)}_s(x,y)p^{(1)}_r(z,w)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}z. \tag{5.30}$$

Considering the operators on positive functions we have, from (5.6),

$$\begin{aligned}U_1(u)U_1(v) &= (\gamma_1U_{11}(u)+\gamma_2U_{21}(u))(\gamma_1U_{11}(v)+\gamma_2U_{21}(v))\\ &\leqslant (\gamma_1+\gamma_2)^2(U_{11}(u)+U_{21}(u))(U_{11}(v)+U_{21}(v))\\ &\leqslant (\gamma_1+\gamma_2)^2(U_{11}(u)+U_{12}(u)+U_{21}(u)+U_{22}(u))\\ &\quad\times(U_{11}(v)+U_{12}(v)+U_{21}(v)+U_{22}(v))\end{aligned}$$

and by (5.10),

$$\left|U_1(u)U_1(v)(\varphi(x)f_\epsilon(x-z)\int \varphi(y)g_\delta(y-w)p^{(1)}_s(x,y)p^{(1)}_r(z,w)\,\mathrm{d}w\,\mathrm{d}y\right|$$
$$\leqslant(\gamma_1+\gamma_2)^2A^{(1)}_{\epsilon,\delta}(x,z),$$

where

$$A^{(1)}_{\epsilon,\delta}(x,z)=\int p^{(1)}_u(x,x')p^{(1)}_v(z,z')|\varphi(x')|f_\epsilon(x'-z')$$
$$\times|\varphi(y)|g_\delta(y-w)p^{(1)}_s(x',y)p^{(1)}_r(z',w)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x'\,\mathrm{d}z',$$

plus three other similar terms involving $p^{(2)}$.

Assume $r,s,u,v>0$ (the other cases are treated similarly).

For the integral on y,w we have

$$\int |\varphi(y)|g_\delta(y-w)p^{(1)}_s(x',y)p^{(1)}_r(z',w)\,\mathrm{d}y\,\mathrm{d}w$$
$$\leqslant||\varphi||_\infty||p^{(1)}_s||_\infty\int g_\delta(y-w)\,\mathrm{d}y\int p^{(1)}_r(z',w)\,\mathrm{d}w$$
$$=||\varphi||_\infty||p^{(1)}_s||_\infty.$$

Hence,

$$A^{(1)}_{\epsilon,\delta}(x,z)\leqslant||\varphi||_\infty||p^{(1)}_s||_\infty\int p^{(1)}_u(x,x')p^{(1)}_v(z,z')|\varphi(x')|f_\epsilon(x'-z')\,\mathrm{d}x'\,\mathrm{d}z.$$

By the definition of f_ϵ and the fact that f has compact support we have, for some $M>0$,

$$A^{(1)}_{\epsilon,\delta}(x,z)\leqslant||\varphi||_\infty||p_s^{(1)}||_\infty||f||_\infty\epsilon^{-d}\int_{|(x'-z')/\epsilon|\leqslant M} p_u^{(1)}(x,x')p_v^{(1)}(z,z')|\varphi(x')|\,\mathrm{d}x'\,\mathrm{d}z'$$

and making the change of variable $z'=\epsilon y$,

$$A^{(1)}_{\epsilon,\delta}(x,z)\leqslant c_2\int p_u^{(1)}(x,x')|\varphi(x')|\left[\int_{|y-x'/\epsilon|\leqslant M} p_v^{(1)}(z,\epsilon y)\,\mathrm{d}y\right]\mathrm{d}x',$$

where $c_2=||\varphi||_\infty||p_s^{(1)}||_\infty||f||_\infty$.

The integral in brackets in the last inequality is bounded as follows (assuming $\epsilon\leqslant 1$):

$$\int_{|y-x'/\epsilon|\leqslant M} p_v^{(1)}(z,\epsilon y)\,\mathrm{d}y\leqslant \sup_{w:|w-x'|\leqslant M} p_v^{(1)}(z,w)\int_{|y-x'/\epsilon|\leqslant M}\mathrm{d}y$$

$$\leqslant (2M)^d\sup_{w:|w-x'|\leqslant M} p_v^{(1)}(z,w).$$

Hence, denoting

$$\tilde{p}_v^{(1)}(z,x')=\sup_{w:|w-x'|\leqslant M} p_v^{(1)}(z,w),$$

we have obtained

$$A^{(1)}_{\epsilon,\delta}(x,z)\leqslant c_3M^dG(x,z),$$

where

$$G(x,z)=\int|\varphi(x')|\,p_u^{(1)}(x,x')\,\tilde{p}_v^{(1)}(z,x')\,\mathrm{d}x'.$$

We will show that $\int G(x,z)\,\mathrm{d}x\,\mathrm{d}z<\infty$.

By Lemma 5.9(b),

$$\int G(x,z)\,\mathrm{d}x\leqslant\int p_u^{(1)}(y)\,\mathrm{d}y\int|\varphi(x)|\,\tilde{p}_v^{(1)}(z,x)\,\mathrm{d}x$$

$$=\int|\varphi(x)|\,\tilde{p}_v^{(1)}(z,x)\,\mathrm{d}x.$$

Hence,

$$\int G(x,z)\,\mathrm{d}x\,\mathrm{d}z\leqslant\int|\varphi(x)|\int\tilde{p}_v^{(1)}(z,x)\,\mathrm{d}z\,\mathrm{d}x$$

$$=\int|\varphi(x)|\int\sup\nolimits_{w:|w-x|\leqslant M}\ p_v^{(1)}(z,w)\,\mathrm{d}z\,\mathrm{d}x.$$

Therefore, it suffices to prove that for fixed $t>0$,

$$\sup_x\int\sup\nolimits_{w:|w-x|\leqslant M}\ p_t^{(1)}(z,w)\,\mathrm{d}z<\infty.$$

We will do it assuming $d=1$ for simplicity (the proof is analogous for $d>1$).

We have

$$\sup_{w:|w-x|\leqslant M} p_t^{(1)}(z,w) = \sup_{w:|w-x|\leqslant M} p_t^{(1)}(z-x-(w-x))$$

$$= \sup_{|h|\leqslant M} p_t^{(1)}(z-x+h).$$

We know that $p_t^{(1)}(\cdot)$ is symmetric and unimodal (Gorostiza et al., 1972, p. 325). Hence,

$$\sup_{|h|\leqslant M} p_t^{(1)}(z-x+h) = p_t^{(1)}(z-x-M) \quad \text{if } z-x-M>0$$

and

$$\sup_{|h|\leqslant M} p_t^{(1)}(z-x+h) = p_t^{(1)}(0) \quad \text{if } z-x\geqslant 0 \quad \text{and} \quad z-x-M\leqslant 0.$$

Then

$$\int \sup\nolimits_{w:|w-x|\leqslant M}\ p_t^{(1)}(z,w)\,\mathrm{d}z$$

$$= \int \sup\nolimits_{|h|\leqslant M}\ p_t^{(1)}(z-x+h)\,\mathrm{d}z$$

$$\leqslant c_4\left(p_t^{(1)}(0)\int_{|z-x|\leqslant M} \mathrm{d}z + \int_{|z-x|>M} p_t^{(1)}(z-x-M)\,\mathrm{d}z\right)$$

$$\leqslant c_5\left(p_t^{(1)}(0) + \int p_t^{(1)}(z-x-M)\,\mathrm{d}z\right)$$

$$\leqslant c_5(p_t^{(1)}(0)+1)$$

and the desired result follows.

So, $\lim_{\epsilon,\delta\to 0}$ can be taken inside the integral in expressions of the form (5.30).

Now, we will show that the limit

$$\lim_{\epsilon,\delta\to 0} U_1(s)U_1(r)\varphi(x)f_\epsilon(x-z)\int \varphi(y)g_\delta(y-w)p_{u-s}^{(1)}(x,y)p_{v-r}^{(1)}(z,w)\,\mathrm{d}y\,\mathrm{d}w$$

in (5.29) exists.

From (5.7), we have

$$U_1(s)U_1(r)\varphi(x)f_\epsilon(x-z)\varphi(y)g_\delta(y-w)p_{u-s}^{(1)}(x,y)p_{v-r}^{(1)}(z,w)\,\mathrm{d}y\,\mathrm{d}w$$

$$= \int \varphi(x')f_\epsilon(x'-z')\varphi(y)g_\delta(y-w)p_{u-s}^{(1)}(x',y)p_{v-r}^{(1)}(z',w)h_r^{(1)}(x,x')h_s^{(1)}(z,z')$$

$$\times \mathrm{d}y\,\mathrm{d}w\,\mathrm{d}x'\,\mathrm{d}z'$$

$$= \int \left[\int\!\!\int f_\epsilon(x'-z')g_\delta(y-w)\varphi(x')\varphi(y)p_{u-s}^{(1)}(x',y)h_r^{(1)}(x,x')\,\mathrm{d}x'\,\mathrm{d}y\right]$$

$$\times p_{v-r}^{(1)}(z',w)h_s^{(1)}(z,z')\,\mathrm{d}w\,\mathrm{d}z'.$$

Since $f_\epsilon(x'-z')g_\delta(y-w)$ approximates the Dirac delta in $L^1(\mathbb{R}^{2d})$ (e.g., Hewitt and Stromberg, 1965, Theorem 21.37), the expression in brackets tends to

$$\varphi(z')\varphi(w)p^{(1)}_{u-s}(z',w)h^{(1)}_r(x,z')$$

in $L^1(\mathbb{R}^{2d})$ as $\epsilon,\ \delta\to 0$, for each x. This implies that the limit (5.29) exists and is given by

$$\lim_{\epsilon,\delta\to 0} U_1(s)U_1(r)\varphi(x)f_\epsilon(x-z)\int \varphi(y)g_\delta(y-w)p^{(1)}_{u-s}(x,y)p^{(1)}_{v-r}(z,w)\,\mathrm{d}y\,\mathrm{d}w$$
$$=\int \varphi(z')\varphi(w)p^{(1)}_{u-s}(z',w)p^{(1)}_{v-r}(z',w)h^{(1)}_r(x,z')h^{(1)}_s(z,z')\,\mathrm{d}w\,\mathrm{d}z'.$$

The other terms in (5.15) are treated similarly.

Thus we have existence of $\lim_{\epsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})$. The proof is analogous for $J^{(2)}$. □

Next, we will establish inequalities between the covariances K and between the functions J for the stationary and general case.

Lemma 5.11. (a) *There exist positive constants c_1 and c_2 such that for any positive $\varphi,\psi\in\mathscr{S}(\mathbb{R}^d)$,*

$$c_1\overline{K}(s,\varphi;t,\psi)\leqslant K(s,\varphi;t,\psi)\leqslant c_2\overline{K}(s,\varphi;t,\psi).$$

(b) *There exist positive constants b_1 and b_2 such that for $\Phi^{(1)},\Phi^{(2)}\in\mathscr{S}(\mathbb{R}^{2d})$,*

$$b_1\overline{J}^{(i)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})\leqslant J^{(i)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})\leqslant b_2\overline{J}^{(i)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)}),\quad i=1,2.$$

Proof. (a) Let $\Gamma=(\gamma_1\lambda,\gamma_2\lambda)$ with $\gamma_1,\gamma_2\geqslant 0$, and $\Phi=(\varphi,\psi)$. Then

$$\begin{aligned}\langle\Gamma,\Phi\rangle &= \gamma_1\langle\lambda,\varphi\rangle+\gamma_2\langle\lambda,\varphi\rangle\\ &=\overline{\gamma}_1(\gamma_1/\overline{\gamma}_1)\langle\lambda,\varphi\rangle+\overline{\gamma}_2(\gamma_2/\overline{\gamma}_2)\langle\lambda,\psi\rangle\\ &\geqslant \min\{\gamma_1/\overline{\gamma}_1,\gamma_2/\overline{\gamma}_2\}(\overline{\gamma}_1\langle\lambda,\varphi\rangle+\overline{\gamma}_2\langle\lambda,\psi\rangle)\\ &= c_1\langle\overline{\Gamma},\Phi\rangle,\end{aligned}$$

where $\overline{\Gamma}$ is the invariant measure at $c_1=\min\{\gamma_1/\overline{\gamma}_1,\gamma_2/\overline{\gamma}_2\}$. Hence we have, from (3.1), with $\Phi^{(1)}=(\varphi,\varphi)$, $\Phi^{(2)}=(\psi,\psi)$, and invariance of $\overline{\Gamma}$ for $U(t)$,

$$\begin{aligned}K(s,\varphi;t,\psi) &= \langle\Gamma,U(s)\ (\Phi^{(1)}\odot U(t-s)\Phi^{(2)})\rangle\\ &\geqslant c_1\langle\overline{\Gamma},U(s)\ (\Phi^{(1)}\odot U(t-s)\Phi^{(2)})\rangle\\ &= c_1\langle\overline{\Gamma},\Phi^{(1)}\odot U(t-s)\Phi^{(2)}\rangle\\ &= c_1\overline{K}(s,\varphi,t,\psi),\end{aligned}$$

which yields the first inequality.

Similarly,

$$\langle\Gamma,\Phi\rangle\leqslant c_2\langle\overline{\Gamma},\Phi\rangle,$$

where $c_2=\max\{\gamma_1/\overline{\gamma}_1,\gamma_2/\overline{\gamma}_2\}$, and the second inequality is proved the same way.

(b) Let $\Phi^{(1)}$ and $\Phi^{(2)}$ be of the form $\Phi^{(1)} = \sum_{i=1}^{n} \varphi_i^{(1)} \otimes \psi_i^{(1)}$, $\Phi^{(2)} = \sum_{j=1}^{m} \varphi_j^{(2)} \otimes \psi_j^{(2)}$. Then from (2.2) and part (a) of the lemma,

$$\begin{aligned} J^{(1)}_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) &= \sum_{i=1}^{n}\sum_{j=1}^{m} K(s, \varphi_i^{(1)}; u, \varphi_j^{(2)}) K(r, \psi_i^{(1)}; v, \psi_j^{(2)}) \\ &\geqslant c_1^2 \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{K}(s, \varphi_i^{(1)}; u, \varphi_j^{(2)}) \overline{K}(r, \psi_i^{(1)}; v, \psi_j^{(2)}) \\ &= c_1^2 \overline{J}^{(1)}_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}). \end{aligned}$$

Since the linear space of elements of the form $\Phi = \sum_{i=1}^{n} \varphi_i \otimes \psi_i$ is dense in $\mathscr{S}(\mathbb{R}^{2d})$, the first inequality for $J^{(1)}$ follows from the continuity of $\overline{J}^{(1)}$ and $J^{(1)}$ (from Lemma 5.6). The other inequality for $J^{(1)}$ is proved similarly.

The proof for $J^{(2)}$ is similar. $\square$

So far we have shown that conditions (i) and (ii) of Theorem 2.2 are satisfied.

We now proceed to verify the remaining conditions of Theorem 2.2 to complete the proof of Theorem 4.1(a). We assume $\alpha_1 \leqslant \alpha_2$.

Case $d \geqslant 2\alpha_1$: Let $\varphi > 0$. By Lemma 5.11, and using (5.14), (5.17), (5.26), (5.27) and Fatou's lemma, we obtain

$$\begin{aligned} &\liminf_{\epsilon\to 0} \int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \\ &\quad\geqslant b_1 \liminf_{\epsilon\to 0} \int_{[0,1]^4} \overline{J}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \\ &\quad\geqslant b_1 \overline{\gamma}_1^2 \left(\int_{[0,1]^4} \int p^{(1)}_{|u-s|}(x,y) p^{(1)}_{|v-r|}(x,y) \varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right. \\ &\qquad \left. + \int_{[0,1]^4} \int p^{(1)}_{|u-r|}(x,y) p^{(1)}_{|v-s|}(x,y) \varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right). \end{aligned}$$

Then by (5.23),

$$\begin{aligned} &\liminf_{\epsilon\to 0} \int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \\ &\quad\geqslant a^2 b_1 \overline{\gamma}_1^2 \left(\int_{[0,1]^4} \int q^{(1)}_{|u-s|}(x,y) q^{(1)}_{|v-r|}(x,y) \varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right. \\ &\qquad \left. + \int_{[0,1]^4} \int q^{(1)}_{|u-r|}(x,y) q^{(1)}_{|v-s|}(x,y) \varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right) \\ &\quad\geqslant C(\varphi) \left(\int (|u-s| + |v-r|)^{-d/\alpha_1}\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right. \\ &\qquad \left. + \int (|u-r| + |v-s|)^{-d/\alpha_1}\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \right), \end{aligned}$$

where the last inequality is due to Lemma 6.2.3 of Bojdecki and Gorostiza (1995). By Lemma 2.6(a) of Bojdecki and Gorostiza (1995) the last integrals diverge for $d \geqslant 2\alpha_1$.

Hence, by part 2 of Theorem 2.2, X does not have SILT for $d \geqslant 2\alpha_1$.

Case $d < 2\alpha_1$: We will verify conditions (iii) and (iv) of Theorem 2.2.

From (5.15) we have

$$|J^{(1)}_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})| \leqslant J^{(1)}_{s,r,u,v}(|\Phi^{(1)}|,|\Phi^{(2)}|)$$

and from Lemma 5.11(b) and the definition of $\Phi^f_{\epsilon,\varphi}$

$$J^{(1)}_{s,r,u,v}(|\Phi^f_{\epsilon,\varphi}|,|\Phi^g_{\delta,\varphi}|) \leqslant b_2 \overline{J}^{(1)}_{s,r,u,v}(|\Phi^f_{\epsilon,\varphi}|,|\Phi^g_{\delta,\varphi}|) = b_2 \overline{J}^{(1)}_{s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}).$$

Therefore it suffices to bound the terms of $\overline{J}^{(1)}$ given in (5.18). For example, for the first one (omitting $\overline{\gamma}_1^2$), i.e.,

$$\overline{J}^{(1,1,1)}_{s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}) = \int |\varphi(x)||\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) p^{(1)}_{|u-s|}(x,y) p^{(1)}_{|v-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w, \tag{5.31}$$

we have, by Lemma 5.9(c) and denoting $\mu_t \equiv \mu_t^{(1)}$,

$$\begin{aligned}
&\overline{J}^{(1,1,1)}_{s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}) \\
&\quad\leqslant \int |\varphi(x)||\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) q^{(1)}_{|u-s|}(x,y) q^{(1)}_{|v-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \\
&\qquad + \int_0^{|u-s|}\int |\varphi(x)||\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) q^{(1)}_m(x,x') \\
&\qquad \times q^{(2)}_{|u-s|-m}(x',y) q^{(1)}_{|v-r|}(z,w)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}x'\,\mu_{|u-s|}(\mathrm{d}m) \\
&\qquad + \int_0^{|v-r|}\int |\varphi(x)||\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) q^{(1)}_m(z,w') \\
&\qquad \times q^{(2)}_{|v-r|-m}(w',w) q^{(1)}_{|u-s|}(x,y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}w'\,\mu_{|v-r|}(\mathrm{d}m) \\
&\qquad + \int_0^{|u-s|}\int_0^{|v-r|}\int |\varphi(x)||\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) q^{(1)}_m(x,x') q^{(2)}_{|u-s|-m}(x',y) \\
&\qquad \times q^{(1)}_n(z,w') q^{(2)}_{|v-r|-n}(w',w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}x'\,\mu_{|u-s|}(\mathrm{d}m)\mu_{|v-r|}(\mathrm{d}n) \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1 we have, by Lemma 6.2.1(c) of Bojdecki and Gorostiza (1995) (using the Chapman–Kolmogorov formula),

$$I_1 \leqslant C(\varphi) H_1(s,r,u,v),$$

where

$$H_1(s,r,u,v) = (|u-s| + |v-r|)^{-d/\alpha_1}.$$

For I_2 we have

$$\begin{aligned}
I_2 \leqslant \sup_x |\varphi(x)| &\int_0^{|u-s|}\int |\varphi(y)| f_\epsilon(x-z) g_\delta(y-w) q^{(1)}_m(x,x') \\
&\times q^{(2)}_{|u-s|-m}(x',y) q^{(1)}_{|v-r|}(z,w)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}x'\,\mu_{|u-s|}(\mathrm{d}m).
\end{aligned}$$

Changing variables $z \to x - z$, and using the Chapman–Kolmogorov formula for $q^{(1)}$ and the self-similarity of $q^{(1)}$,

$$I_2 \leqslant \sup_x |\varphi(x)| \int_0^{|u-s|} \int |\varphi(y)| f_\epsilon(z) g_\delta(y-w) q^{(1)}_{|v-r|}(x, w+z) q^{(1)}_m(x, x')$$

$$\times q^{(2)}_{|u-s|-m}(x', y) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}x'\, \mu_{|u-s|}(\mathrm{d}m)$$

$$= \sup_x |\varphi(x)| \int_0^{|u-s|} \int |\varphi(y)| f_\epsilon(z) g_\delta(y-w) q^{(1)}_{m+|v-r|}(x', w+z)$$

$$\times q^{(2)}_{|u-s|-m}(x', y) \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}x'\, \mu_{|u-s|}(\mathrm{d}m)$$

$$\leqslant c_1 \sup_x |\varphi(x)| \int_0^{|u-s|} \int |\varphi(y)| (m + |v-r|)^{-d/\alpha_1} f_\epsilon(z) g_\delta(y-w)$$

$$\times q^{(2)}_{|u-s|-m}(x', y) \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}x' \mu_{|u-s|}(\mathrm{d}m)$$

$$= c_1 \sup_x |\varphi(x)| \int |\varphi(y)|\, \mathrm{d}y \int_0^{|u-s|} (m + |v-r|)^{-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m)$$

$$\leqslant C(\varphi) H_2(s, r, u, v),$$

where

$$H_2(s, r, u, v) = \int_0^{|u-s|} (m + |v-r|)^{-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m).$$

In a similar way, we obtain for I_3

$$I_3 \leqslant C(\varphi) H_3(s, r, u, v),$$

where

$$H_3(s, r, u, v) = \int_0^{|v-r|} (m + |u-s|)^{-d/\alpha_1} \mu_{|v-r|}(\mathrm{d}m).$$

For I_4 we have, similarly,

$$I_4 \leqslant \sup_x |\varphi(x)| \int_0^{|u-s|} \int_0^{|v-r|} \int |\varphi(y)| f_\epsilon(z) g_\delta(y-w) q^{(1)}_m(x, x') q^{(2)}_{|u-s|-m}(x', y)$$

$$\times q^{(1)}_n(x, w'+z) q^{(2)}_{|v-r|-n}(w', w)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}w'\, \mathrm{d}x'\, \mu_{|u-s|}(\mathrm{d}m) \mu_{|v-r|}(\mathrm{d}n)$$

$$\leqslant \sup_x |\varphi(x)| \int_0^{|u-s|} \int_0^{|v-r|} \int |\varphi(y)| f_\epsilon(z) g_\delta(y-w) q^{(1)}_{m+n}(x', w'+z)$$

$$\times q^{(2)}_{|u-s|-m}(x', y) q^{(2)}_{|v-r|-n}(w', w)\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}w'\, \mathrm{d}x' \mu_{|u-s|}(\mathrm{d}m) \mu_{|v-r|}(\mathrm{d}n)$$

$$\leqslant c_1 \sup_x |\varphi(x)| \int_0^{|u-s|} \int_0^{|v-r|} (m+n)^{-d/\alpha_1} \int |\varphi(y)| f_\epsilon(z) g_\delta(y-w)$$

$$\times q^{(2)}_{|u-s|-m}(x', y) q^{(2)}_{|v-r|-n}(w', w)\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}w'\, \mathrm{d}x' \mu_{|u-s|}(\mathrm{d}m) \mu_{|v-r|}(\mathrm{d}n)$$

$$= c_1 \sup_x |\varphi(x)| \int |\varphi(y)|\,\mathrm{d}y \int_0^{|u-s|} \int_0^{|v-r|} (m+n)^{-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m)\mu_{|v-r|}(\mathrm{d}n)$$
$$\leqslant C(\varphi) H_4(s,r,u,v),$$

where

$$H_4(s,r,u,v) = \int_0^{|u-s|} \int_0^{|v-r|} (m+n)^{-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m)\mu_{|v-r|}(\mathrm{d}n).$$

We proceed analogously with the other terms of $\overline{J}^{(1)}$ in Eq. (5.18). For the terms which contain $p^{(1)}$ and $p^{(2)}$ we cannot use the Chapman–Kolmogorov formula and self-similarity as we did above, but we make the same changes of variables and employ Lemma 5.9(d).

We do the same thing for $\overline{J}^{(2)}$, noting that when the self-similarity of $q^{(2)}$ is used, we obtain a bound of the form t^{-d/α_2}, $t \in [0,2]$, but $t^{-d/\alpha_2} \leqslant Ct^{-d/\alpha_1}$ since $\alpha_1 \leqslant \alpha_2$ (as in the proof of Lemma 5.9(d)).

In conclusion, the function

$$G_\varphi(s,r,u,v) = C(\varphi) \sum_{i=1}^{4} (H_i(s,r,u,v) + H_i(s,r,v,u))$$

is a bound for $\overline{J}_{s,r,u,v}(\Phi^f_{\epsilon,\varphi}, \Phi^g_{\delta,\varphi})$.

By Lemma 2.6(a) of Bojdecki and Gorostiza (1995), H_1 is integrable on $[0,1]^4$ for $d < 2\alpha_1$. We will show that H_2, H_3, H_4 are integrable as well.

For $d \neq \alpha_1$,

$$\begin{aligned}
\int_{[0,1]^2} H_2(s,r,u,v)\,\mathrm{d}v\,\mathrm{d}r &= \int_{[0,1]^2} \int_0^{|u-s|} (m+|v-r|)^{-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m)\,\mathrm{d}v\,\mathrm{d}r \\
&= 2\int_0^{|u-s|} \int_0^1 \int_0^v (m+r)^{-d/\alpha_1}\,\mathrm{d}r\,\mathrm{d}v\,\mu_{|u-s|}(\mathrm{d}m) \\
&= 2\int_0^{|u-s|} \int_0^1 \frac{1}{1-d/\alpha_1}((m+v)^{1-d/\alpha_1} - m^{1-d/\alpha_1})\,\mathrm{d}v \\
&\quad \times \mu_{|u-s|}(\mathrm{d}m) \\
&= 2\int_0^{|u-s|} \left(\frac{1}{(2-d/\alpha_1)(1-d/\alpha_1)} \right. \\
&\quad \left. \times ((m+1)^{2-d/\alpha_1} - m^{2-d/\alpha_1}) - \frac{m^{1-d/\alpha_1}}{1-d/\alpha_1} \right) \mu_{|u-s|}(\mathrm{d}m) \\
&\leqslant C - \frac{2}{1-d/\alpha_1} \int_0^{|u-s|} m^{1-d/\alpha_1} \mu_{|u-s|}(\mathrm{d}m) \\
&= C - \frac{2}{1-d/\alpha_1} E((\theta^{(1)}_{|u-s|})^{1-d/\alpha_1}),
\end{aligned}$$

where $\theta^{(1)}_{|u-s|}$ is as in Lemma 5.9. The last expression is bounded by C if $d<\alpha_1$. If $d>\alpha_1$, denoting $a=d/\alpha_1-1$ we have

$$\begin{aligned}
E((\theta^{(1)}_{|u-s|})^{-a}) &= \int_0^\infty P\left[\left(\theta^{(1)}_{|u-s|}\right)^{-a} \geqslant x\right] \mathrm{d}x \\
&= \int_0^\infty P\left[\theta^{(1)}_{|u-s|} \leqslant x^{-1/a}\right] \mathrm{d}x \\
&\leqslant \int_0^\infty P[\tau \leqslant x^{-1/a} \wedge |u-s|]\,\mathrm{d}x,
\end{aligned}$$

where τ is exponential (V_1). For any $t>0$,

$$\begin{aligned}
\int_0^\infty P[\tau\leqslant x^{-1/a}\wedge t]\,\mathrm{d}x &= \int_0^\infty (1-\mathrm{e}^{-V_1(x^{-1/a}\wedge t)})\,\mathrm{d}x \\
&= \int_0^{t^{-a}} (1-\mathrm{e}^{-V_1 t})\,\mathrm{d}x + \int_{t^{-a}}^\infty (1-\mathrm{e}^{-V_1 x^{-1/a}})\,\mathrm{d}x \\
&\leqslant (1-\mathrm{e}^{-V_1 t})t^{-a} + V_1\int_{t^{-a}}^\infty x^{-1/a}\,\mathrm{d}x \\
&\leqslant (1-\mathrm{e}^{-V_1 t})t^{1-d/\alpha_1} + V_1\frac{d-\alpha_1}{2\alpha_1-d}t^{2-d/\alpha_1}
\end{aligned}$$

and the last expression is uniformly bounded for $t\in[0,1]$. Hence,

$$\int_{[0,1]^4} H_2(s,r,u,v)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v < \infty\,.$$

For $d=\alpha_1$ we have, for some constant $C_1>0$,

$$\begin{aligned}
\int_{[0,1]^2} H_2(s,r,u,v)\,\mathrm{d}v\,\mathrm{d}r &= 2\int_0^{|u-s|} [(m+1)\log(m+1)-m\log m]\mu_{|u-s|}(\mathrm{d}m) \\
&\leqslant C_1|u-s|.
\end{aligned}$$

Hence,

$$\int_{[0,1]^4} H_2(s,r,u,v)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v < \infty.$$

The proofs are similar for H_3 and H_4.

Therefore condition (iii) of Theorem 2.2 is verified.

For condition (iv) the proof of Lemma 2.6 of Bojdecki and Gorostiza (1995) holds here as well.

The proof of part (a) of Theorem 4.1 is complete.

Now we turn to part (b). The proof is similar to part (a), but now some formulas are simpler. We assume $\alpha_1\leqslant\alpha_2$.

The covariance $K_1(s,\phi;t,\psi)$ of M_1 is obtained from the covariance $K(s,\Phi;t,\Psi)$ of (M_1,M_2) putting $\Phi=(\varphi,0)$, $\Psi=(\psi,0)$. Hence, from (3.1) we obtain

$$K_1(s,\varphi;t,\psi) = \left\langle \begin{pmatrix}\gamma_1\lambda\\ \gamma_2\lambda\end{pmatrix}, \begin{pmatrix}U_{11}(s) & U_{12}(s)\\ U_{21}(s) & U_{22}(s)\end{pmatrix}\left[\begin{pmatrix}\varphi\\ 0\end{pmatrix}\odot\begin{pmatrix}U_{11}(t-s)\psi\\ U_{21}(t-s)\psi\end{pmatrix}\right]\right\rangle$$

$$= \left\langle \begin{pmatrix} \gamma_1 \lambda \\ \gamma_2 \lambda \end{pmatrix}, \begin{pmatrix} U_{11}(s)(\varphi U_{11}(t-s)\psi) \\ U_{21}(s)(\varphi U_{11}(t-s)\psi) \end{pmatrix} \right\rangle$$

$$= \int (\gamma_1 U_{11}(s) + \gamma_2 U_{21}(s))(\varphi U_{11}(t-s)\psi)(x)\,\mathrm{d}x$$

$$= \int U_1(s)(\varphi U_{11}(t-s)\psi)(x)\,\mathrm{d}x$$

$$= \int U_1(s) \left(\varphi(\cdot) \int \psi(y) h_{t-s}^{(11)}(\cdot, y)\,\mathrm{d}y \right)(x)\,\mathrm{d}x. \tag{5.32}$$

Similarly, for the covariance K_2 of M_2 we obtain

$$K_2(s,\varphi;t,\psi) = \int U_2(s) \left(\varphi(\cdot) \int \psi(y) h_{t-s}^{(22)}(\cdot, y)\,\mathrm{d}y \right)(x)\,\mathrm{d}x. \tag{5.33}$$

For the stationary case the covariances are

$$\overline{K}_1(s,\varphi;t,\psi) = \overline{\gamma}_1 \int \varphi(x) \left(\int \psi(y) h_{t-s}^{(11)}(x,y)\,\mathrm{d}y \right)(x)\,\mathrm{d}x,$$

$$\overline{K}_2(s,\varphi;t,\psi) = \overline{\gamma}_2 \int \varphi(x) \left(\int \psi(y) h_{t-s}^{(22)}(x,y)\,\mathrm{d}y \right)(x)\,\mathrm{d}x.$$

The functions $J_{(i),s,r,u,v}$, $i=1,2$, with $s \leqslant r \leqslant u \leqslant v$, for M_1 and M_2 are computed as in Lemma 5.3:

$$\begin{aligned} &J_{(i),s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) \\ &\quad = \int U_i(s)U_i(r) \left(\int \Phi^{(1)}(x,z)\Phi^{(2)}(y,w) h_{u-s}^{(ii)}(x,y) h_{v-r}^{(ii)}(z,w)\,\mathrm{d}y\,\mathrm{d}w \right) \mathrm{d}x\,\mathrm{d}z \\ &\qquad + \int U_i(s)U_i(r) \left(\int \Phi^{(1)}(x,z)\Phi^{(2)}(w,y) h_{v-s}^{(ii)}(x,y) h_{u-r}^{(ii)}(z,w)\,\mathrm{d}y\,\mathrm{d}w \right) \mathrm{d}x\,\mathrm{d}z. \end{aligned} \tag{5.34}$$

From (5.9),

$$h_t^{(ii)}(x,y) \leqslant p_t^{(i)}(x,y), \quad i=1,2. \tag{5.35}$$

Conditions (i) and (ii) of Theorem 2.2 for M_1 and M_2 are proved similarly as Corollary 5.7 and Proposition 5.10 (the proof of Proposition 5.10 can be shortened using the Chapman–Kolmogorov formula).

The equation

$$h_t^{(ii)}(x,y) = \mathrm{e}^{-V_i t} q_t^{(i)}(x,y) + V_i \int_0^t \mathrm{e}^{-V_i s} \int_{R^d} q_s^{(i)}(x,z) h_{t-s}^{(ji)}(z,y)\,\mathrm{d}z\,\mathrm{d}s, \quad i \neq j$$

is proved the same way as Lemma 5.8, whence follow the inequalities

$$h_t^{(ii)}(x,y) \geqslant c_1 q_t^{(i)}(x,y), \tag{5.36}$$

$$h_t^{(ii)}(x,y) \leqslant q_t^{(i)}(x,y) + c_2 \int_0^t \int_{R^d} q_s^{(i)}(x,z) h_{t-s}^{(ji)}(z,y)\,\mathrm{d}z\,\mathrm{d}s, \quad i \neq j, \tag{5.37}$$

where c_1 and c_2 are some positive constants.

For $i=1,2$, suppose $d\geqslant 2\alpha_i$ and let $\varphi>0$. Then, from (5.34), (5.36), and similarly as the proof of part (a) of the theorem, using Lemma 5.11(b) (adapted to the present case), and denoting by $\overline{J}_{(i)}$ the functional $J_{(i)}$ for the stationary case, we have

$$\begin{aligned}
&\liminf_{\epsilon\to 0}\int_{[0,1]^4} J_{(i),s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\\
&\geqslant b_1\liminf_{\epsilon\to 0}\int_{[0,1]^4}\overline{J}_{(i),s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^f_{\epsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\\
&\geqslant c_3\left(\int_{[0,1]^4}\int h^{(ii)}_{|u-s|}(x,y)h^{(ii)}_{|v-r|}(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right.\\
&\quad\left.+\int_{[0,1]^4}\int h^{(ii)}_{|u-r|}(x,y)h^{(ii)}_{|v-s|}(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right)\\
&\geqslant c_4\left(\int_{[0,1]^4}\int q^{(i)}_{|u-s|}(x,y)q^{(i)}_{|v-r|}(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right.\\
&\quad\left.+\int_{[0,1]^4}\int q^{(i)}_{|u-r|}(x,y)q^{(i)}_{|v-s|}(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right)\\
&\geqslant C(\varphi)\left(\int(|u-s|+|v-r|)^{-d/\alpha_i}\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right.\\
&\quad\left.+\int(|u-r|+|v-s|)^{-d/\alpha_i}\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right)\\
&=\infty,
\end{aligned}$$

where we used Lemmas 6.2.3 and 2.6 of Bojdecki and Gorostiza (1995) in the last steps.

Therefore, by part 2 of Theorem 2.2 the SILT of M_i does not exist for $d\geqslant 2\alpha_i$, $i=1,2$.

We now show existence of SILT for M_i when $d<2\alpha_i$, $i=1,2$.

From (5.34) and Lemma 5.11(b) (adapted to this case) we have

$$\begin{aligned}
\left|J_{(i),s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})\right| &\leqslant b_2\overline{J}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|})\\
&=\overline{J}^{(1)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|})+\overline{J}^{(2)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}),
\end{aligned}$$

where

$$\begin{aligned}
\overline{J}^{(1)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}) = b_2\overline{\gamma}_i^{-2}\int & |\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)\\
&\times h^{(ii)}_{|u-s|}(x,y)h^{(ii)}_{|v-r|}(z,w)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w,
\end{aligned}$$

$$\begin{aligned}
\overline{J}^{(2)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|}) = b_2\overline{\gamma}_i^{-2}\int & |\varphi(x)||\varphi(w)|f_\epsilon(x-z)g_\delta(y-w)\\
&\times h^{(ii)}_{|v-s|}(x,y)h^{(ii)}_{|u-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w.
\end{aligned}$$

By inequality (5.35),

$$\overline{J}^{(1)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|})$$
$$\leqslant b_2\overline{\gamma}_i^{\,2}\int|\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)p^{(i)}_{|u-s|}(x,y)p^{(i)}_{|v-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w,$$

$$\overline{J}^{(2)}_{(i),s,r,u,v}(\Phi^f_{\epsilon,|\varphi|},\Phi^g_{\delta,|\varphi|})$$
$$\leqslant b_2\overline{\gamma}_i^{\,2}\int|\varphi(x)||\varphi(w)|f_\epsilon(x-z)g_\delta(y-w)p^{(i)}_{|v-s|}(x,y)p^{(i)}_{|u-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w.$$

Each of the right-hand sides in these two expressions is of the form which appears in part (a); see (5.30). Hence, we can use the result of part (a) to conclude that for $d<2\alpha_1$ the SILT's for M_1 and M_2 exist, and they are continuous processes.

It remains to show existence and continuity of SILT of M_2 for $2\alpha_1\leqslant d<2\alpha_2$. Conditions (i), (ii) and (iv) of Theorem 2.2 are verified similarly as above. Only condition (iii) requires a little more work.

From (5.34), (5.35), (5.37) we have

$$J^{(1)}_{(2),s,r,u,v}(\Phi^f_{\epsilon,\varphi},\Phi^g_{\delta,\varphi})$$
$$\leqslant C\Bigg[\int|\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)q^{(2)}_{|u-s|}(x,y)q^{(2)}_{|v-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w$$
$$+\int_0^{|u-s|}\int|\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)q^{(2)}_m(x,x')$$
$$\times h^{(12)}_{|u-s|-m}(x',y)q^{(2)}_{|v-r|}(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}x'\mathrm{d}m$$
$$+\int_0^{|v-r|}\int|\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)q^{(2)}_m(z,w')$$
$$\times h^{(12)}_{|v-r|-m}(w',w)q^{(2)}_{|u-s|}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}w'\mathrm{d}m$$
$$+\int_0^{|u-s|}\int_0^{|v-r|}\int|\varphi(x)||\varphi(y)|f_\epsilon(x-z)g_\delta(y-w)q^{(2)}_m(x,x')h^{(12)}_{|u-s|-m}(x',y)$$
$$\times q^{(2)}_n(z,w')h^{(12)}_{|v-r|-n}(w',w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}x'\,\mathrm{d}m\,\mathrm{d}n\Bigg].$$

From here on the calculations are analogous to part (a). We use the Chapman–Kolmogorov formula for $q^{(2)}$ and the selfsimilarity of $q^{(2)}$, (5.5), and Lemma 6.2.1(c) of Bojdecki and Gorostiza (1995). We obtain a function $G_\varphi(s,r,u,v)$ which is integrable for $d<2\alpha_2$.

This completes the proof of part (b). □

Acknowledgements

Both authors express their gratitude to CIMAT (Guanajuato), where most of this work was done.

References

Adler, R.J., Feldman, R., Lewin, M., 1991. Intersection local times for infinite systems of Brownian motions and for the Brownian density process. Ann. Probab. 19, 192–220.

Adler, R.J., Rosen, J.S., 1993. Intersection local times of all orders for Brownian and stable density processes – construction, renormalization and limit laws. Ann. Probab. 21, 1073–1123.

Athreya, K.B., Ney, P.E., 1972. Branching Processes. Springer, Berlin.

Bojdecki, T., Gorostiza, L.G., 1995. Self-intersection local time for Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes: existence, path continuity and examples. Stochastic Process Appl. 60, 191–226.

Bojdecki, T., Gorostiza, L.G., 1996. Self-intersection local time for Gaussian $\mathscr{S}'$-processes, and application to fluctuation limits of branching particle systems. Stochastic Analysis: Random Fields and Measure-Valued Processes. Israel Math. Conf. Proc 10, 49–55.

Bojdecki, T., Gorostiza, L.G., 1997. Self-intersection local time for $\mathscr{S}'(\mathbb{R}^d)$-Wiener processes and related Ornstein–Uhlenbeck processes. Preprint.

Fernández, B., 1991. Markov properties of the fluctuation limit of a particle system with death. J. Math. Anal. Appl. 155, 66–77.

Gorostiza, L.G., Roelly, S., Wakolbinger, A., 1992. Persistence of critical multitype particle and measure branching processes. Probab. Theory Related Fields 92, 313–335.

Hewitt, E., Stromberg, K., 1965. Real and Abstract Analysis. Springer, New York.

López-Mimbela, J.A., 1992. Fluctuation limits of multitype branching random fields. J. Multivariate Anal. 40, 56–83.

López-Mimbela, J.A., Wakolbinger, A., 1996. Clumping in multitype-branching trees. Adv. Appl. Probab. 28, 1034–1050.

Lukacs, E., 1970. Characteristic Functions, second ed. Griffin, London.

Pazy, A., 1983. Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, Berlin.

Taylor, S.J., 1966. Multiple points for the sample paths of the symmetric stable process. Z. Wahrsch. Verw. Geb. 5, 247–264.

Treves, F., 1967. Topological Vector Spaces, Distributions and Kernels. Academic Press, New York.